Limits involving $\ln(x)$

We can use the rules of logarithms given above to derive the following information about limits.

$$\lim_{x \to \infty} \ln x = \infty, \qquad \lim_{x \to 0} \ln x = -\infty.$$

- We saw the last day that $\ln 2 > 1/2$.
- Using the rules of logarithms, we see that $\ln 2^m = m \ln 2 > m/2$, for any integer m.
- Because $\ln x$ is an increasing function, we can make $\ln x$ as big as we choose, by choosing x large enough, and thus we have
$$\lim_{x \to \infty} \ln x = \infty.$$
.
- Similarly $\ln\left(\frac{1}{2^n}\right) = -n \ln 2 < -n/2$ and as x approaches 0 the values of $\ln x$ approach $-\infty$.

Example

Find the limit $\lim_{x \to \infty} \ln(\frac{1}{x^2+1})$.

- As $x \to \infty$, we have $\frac{1}{x^2+1} \to 0$
- Letting $u = \frac{1}{x^2+1}$, we have

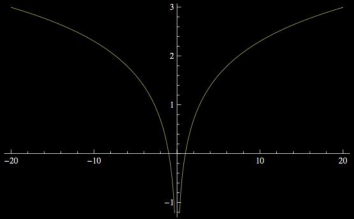
$$\lim_{x \to \infty} \ln(\frac{1}{x^2+1}) = \lim_{u \to 0} \ln(u) = -\infty.$$

Extending the antiderivative of $1/x$

We can extend our antiderivative of $1/x$(the natural logarithm function) to a function with a larger domain by composing $\ln x$ with the absolute value function $|x|$. . We have :

$$\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

This is an even function with graph



We have $ln|x|$ is also an antiderivative of $1/x$ with a larger domain than $ln(x)$.

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\int \frac{1}{x}dx = \ln|x| + C}$$

Using Chain Rule for Differentiation

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}$$

- Example 1: Differentiate $\ln|\sin x|$.
- Using the chain rule, we have

$$\frac{d}{dx}\ln|\sin x| = \frac{1}{(\sin x)}\frac{d}{dx}\sin x$$

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$$= \frac{\cos x}{\sin x}$$

Using Chain Rule for Differentiation : Example 2

Differentiate

$$\ln|\sqrt[3]{x-1}|.$$

- We can simplify this to finding $\frac{d}{dx}\left(\frac{1}{3}\ln|x-1|\right)$, since $\ln|\sqrt[3]{x-1}| = \ln|x-1|^{1/3}$
- $$\frac{d}{dx}\frac{1}{3}\ln|x-1| = \frac{1}{3}\frac{1}{(x-1)}\frac{d}{dx}(x-1) = \frac{1}{3(x-1)}$$

Using Substitution

Reversing our rules of differentiation above, we get:

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C} \quad \text{and} \quad \boxed{\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C}$$

- **Example** Find the integral $\int \frac{x}{3-x^2} dx$
- Using substitution, we let $u = 3 - x^2$.

$$du = -2x\ dx, \qquad x\ dx = \frac{du}{-2},$$

- $$\int \frac{x}{3-x^2} dx = \int \frac{1}{-2(u)}\ du$$
- $$= \frac{-1}{2} \ln|u| + C = \frac{-1}{2} \ln|3 - x^2| + C$$